A Radial Reduction Theorem for Quasilinear PDEs

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Radial Patterns in Fluid Problems

Localised axisymmetric and dihedral patterns form in various fluid problems

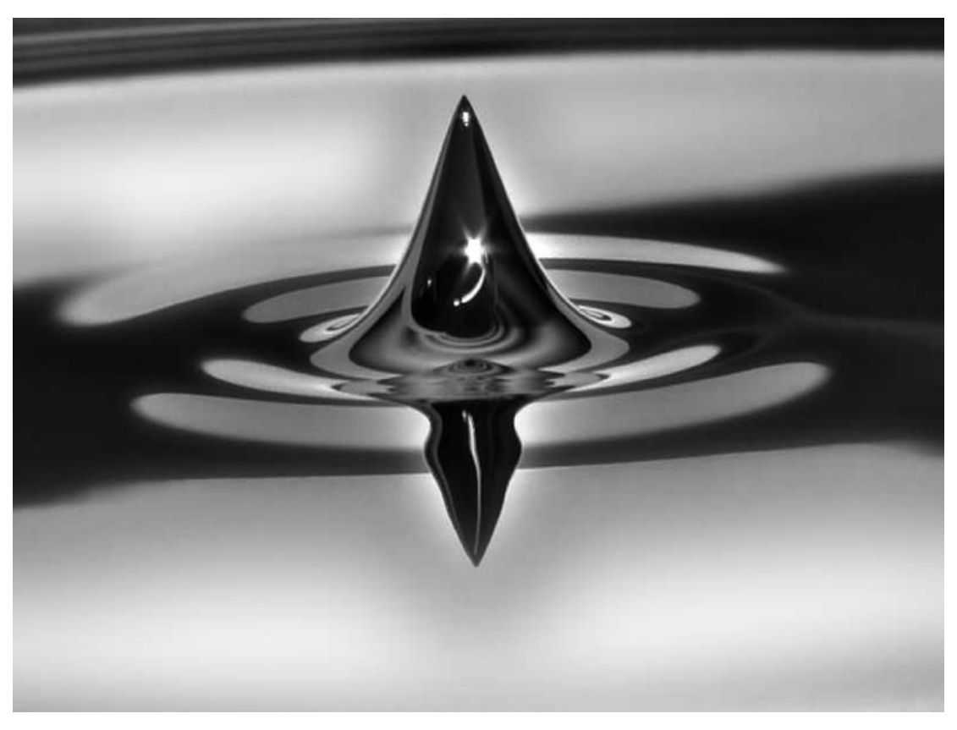
surrounded by a flat state

invariant under continuous and discrete rotations, respectively

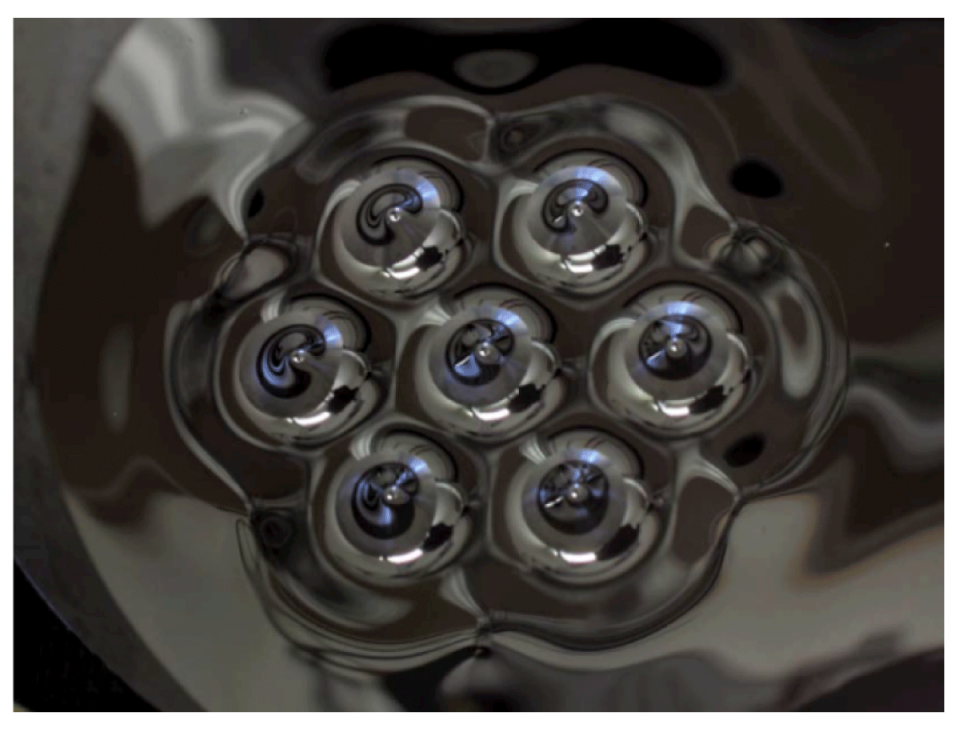
A ferrofluid subject to magnetic effects can exhibit spots and hexagons

magnetic fluid

axisymmetric *dihedral*



Localised ferrofluid spot[1]



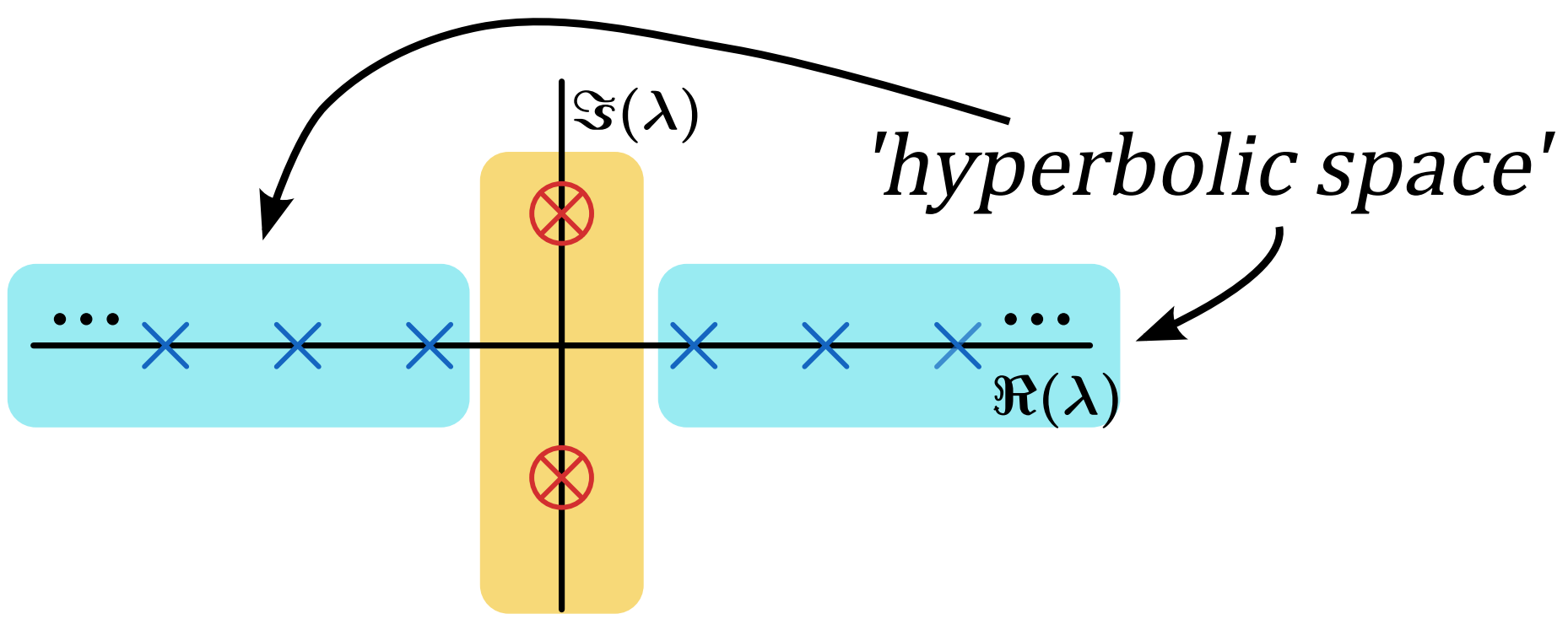
Localised ferrofluid hexagon[2]

Ferrofluid Spots: PDE Formulation[3]

Axisymmetric free surface problem, modelled by a quasilinear partial differential equation

$$\frac{\mathrm{d}}{\mathrm{d}r}u = \underbrace{\mathrm{L}(r)u}_{\text{Linear operator with } \partial_{zz} \text{ terms}} + \underbrace{N(u,\partial_z u;\varepsilon,r)}_{\text{Quasilinearity: } N \text{ has non-smoothing effects}}$$

Linear spectrum: $\lambda u = \mathrm{L}(\infty)u$



'centre space'

Project onto eigenbasis of $\mathrm{L}(\infty)$, with u_c and u_h centre and hyperbolic variables

$$\underbrace{\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}u_c &= \mathrm{M}_{\mathrm{c}}(r)u_c + \mathrm{F}_{\mathrm{c}}(u_c,u_h;\varepsilon,r) \\ \frac{\mathrm{d}}{\mathrm{d}r}u_h &= \mathrm{M}_{\mathrm{h}}(r)u_h + \mathrm{F}_{\mathrm{h}}(u_c,u_h;\varepsilon,r) \end{aligned}}_{\text{infinite dimensional ODE}} \tag{1}$$

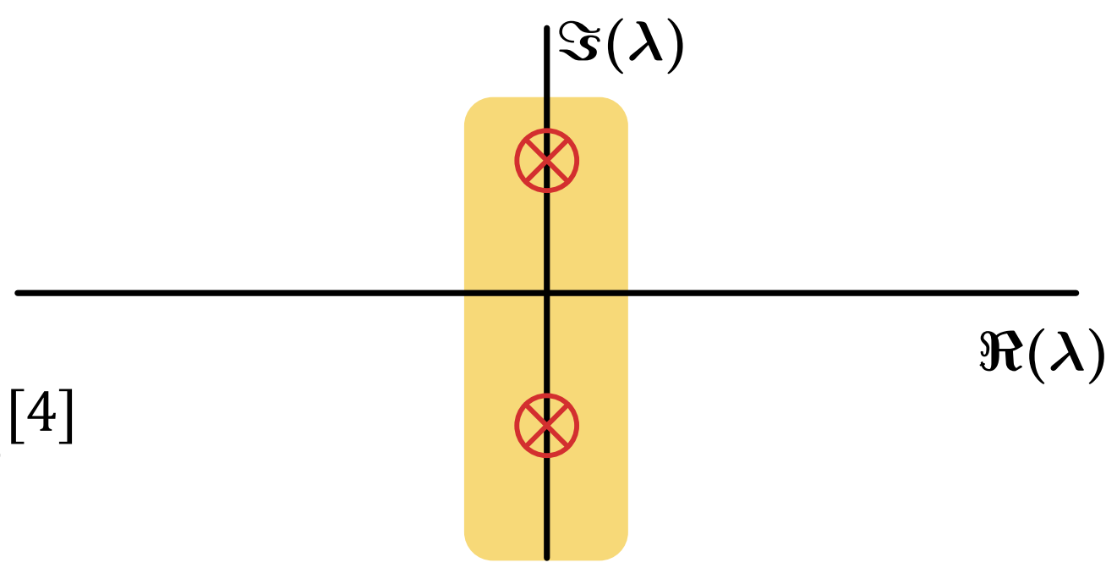
Spots in the Swift-Hohenberg Equation (SHE)

Swift-Hohenberg equation: $0 = -(1+\Delta)^2 u - \varepsilon u + \nu u^2 - u^3$

Assume axisymmetry to obtain semilinear ordinary differential equation

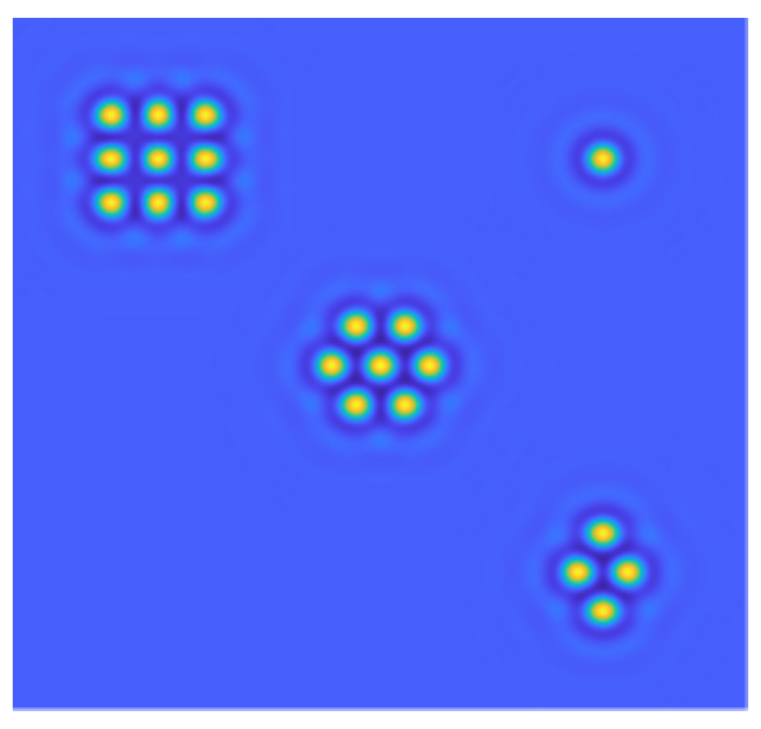
$$\frac{\mathrm{d}}{\mathrm{d}r}u = \mathrm{M}_{\mathrm{c}}(r)u + \mathrm{N}(u;\varepsilon,r) \tag{2}$$

Linear Spectrum: $\lambda u = \mathrm{M}_{\mathrm{c}}(\infty)u$



2008: Numerical study of localised hexagons[4]

2009: Proof of localised spots using radial spatial dynamics[5]



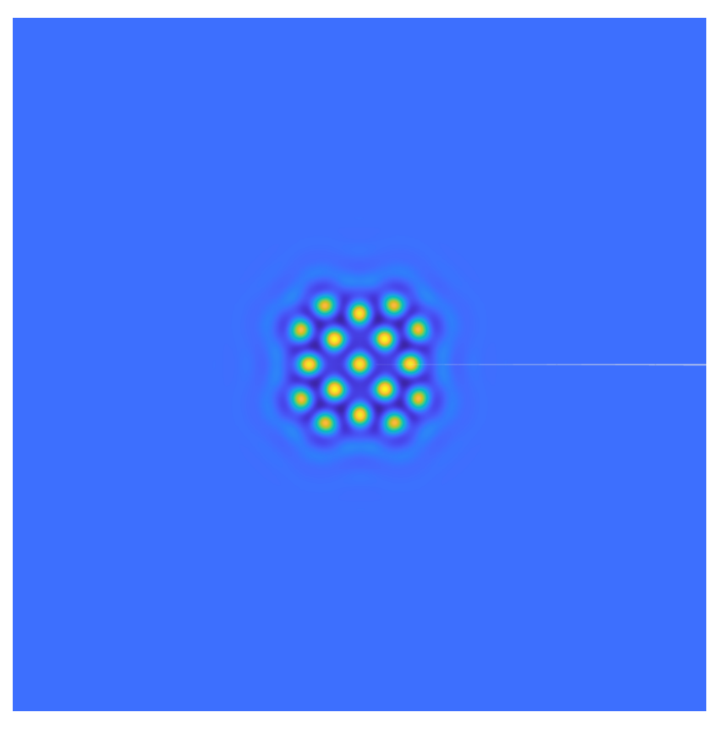
Localised spots and dihedral patterns in the SHE[6]

Localised Dihedral Patterns in the SHE

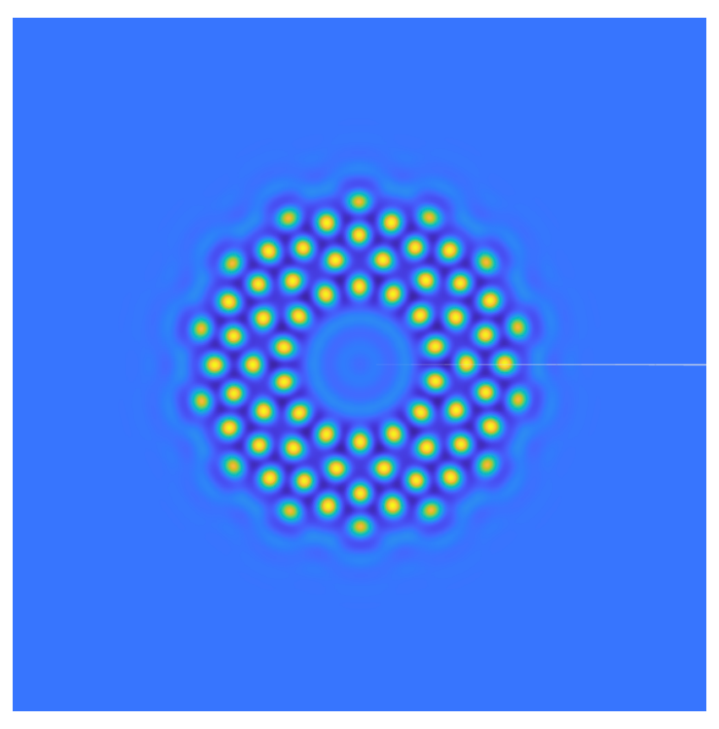
Consider dihedral groups D_m, for arbitrary m ∈ ℕ

invariant under rotations of 2π/m

Proof of localised dihedral patterns in Galerkin model[6]



Localised D_4 Pattern



Localised D_{14} Pattern

Aims

Develop a radial reduction theorem for quasilinear PDEs of the form (1)

Prove the existence of localised spots in fluid problems

Existence of localised dihedral patterns

Plan and Challenges

Extend quasilinear centre-manifold reduction[7] from 1D to radial PDEs

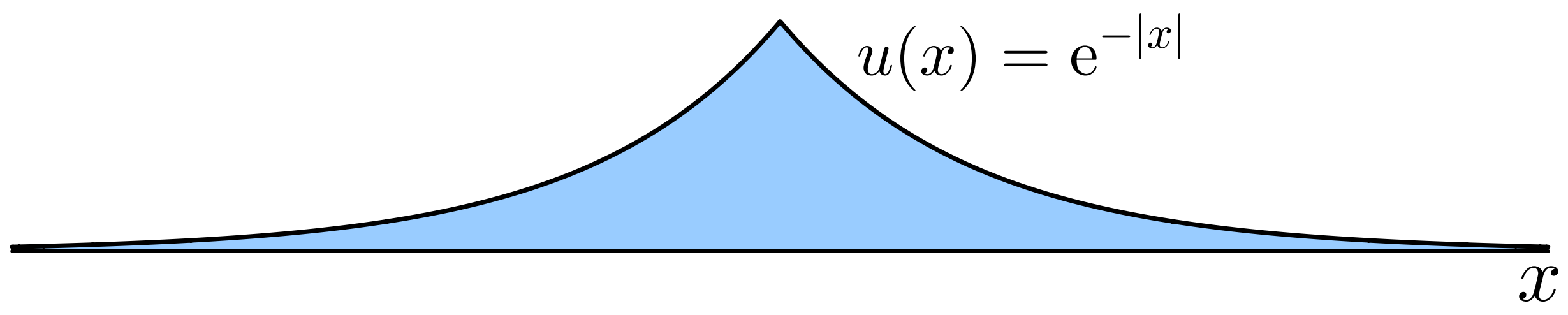
Need to prove Maximal Regularity results for (1):

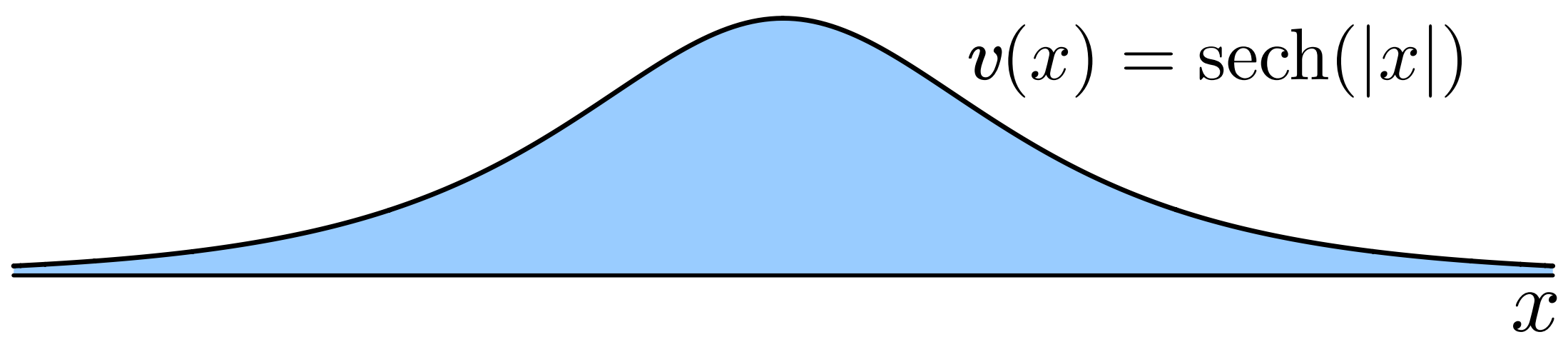
The operator $M_h(r)$ provides additional smoothing to balance the quasilinear nonlinearity $F_h(u_c, u_h; \varepsilon, r)$

Additional restrictions on function spaces for solutions, including conditions at *r=0*

Planar vs. Radial Smoothness

Radially smooth function $u(|x|) = e^{-|x|}$ vs. Planar smooth function $v(|x|) = \mathrm{sech}(|x|)$





Is $v'(|x|)$ a planar smooth function?

Requires a novel framework of radial function spaces with non-autonomous differential operators

References

[1] R. Richter, *Europhysics News*, 42.3 (2011)
[2] D. Lloyd et al., *J. Fluid Mech.*, 783 (2015)
[3] D. Hill et al., *J. Nonlinear Sci.*, 31 (2021)
[4] D. Lloyd et al., *SIADS*, 7 (2008)
[5] D. Lloyd et al., *Nonlinearity*, 22 (2009)
[6] D. Hill et al., *Nonlinearity*, 36 (2023)
[7] A. Mielke, *Math. Meth. Appl. Sci.*, 10 (1988)