A Polar Approach to Fully Localised Planar Patterns

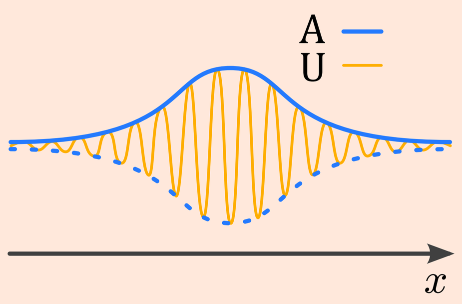
Dan J. Hill

Universität des Saarlandes

UNTERSTÜTZT VON / SUPPORTED BY Alexander von HUMBOLDT STIFTUNG

Planar Localisation

1D



Existence is well understood

Analysis via spatial dynamics

Critical wave number k - 'Fast' oscillations

Exponentially localised envelope A - 'Slow' decay

2D

Most physical patterns appear in 2D (or 3D) settings

Localisation in 2 dimensions - no 'time-like' variable for spatial dynamics

Continuum of bifurcating wave numbers $|\mathbf{k}| = k$

Localised envelope over periodic domain-covering pattern?

Localised hexagon on the surface of a ferrofluid [LGR+15]

Key Problem

How can we study localised 2D patterns without a spatial dynamics viewpoint?

Can we prove the existence of fully localised patterns in the Swift-Hohenberg equation?

$$\partial_t u = -(1+\Delta)^2 u - \mu u + \nu u^2 - u^3 \quad (1)$$

Radial Spatial Dynamics

A simple example: Localised axisymmetric patterns (*Invariant under continuous rotations*)

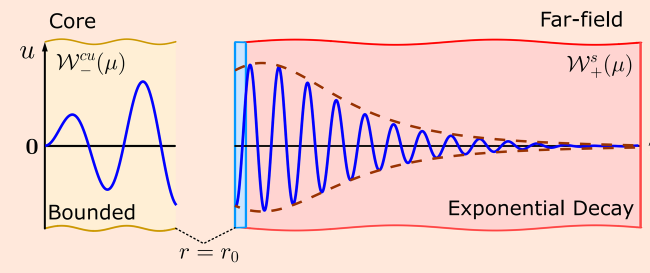
$u(t,r,\theta) = u_0(r)$ - Reduces (1) to a radial ODE

2003 - A. Scheel, *AMS Mem.* [Sch03]

Designed for domain-covering patterns

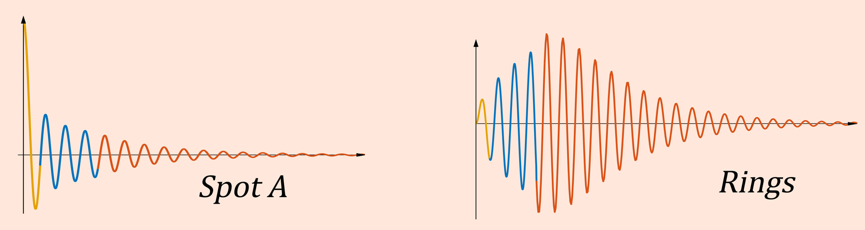
Construct local invariant manifolds

Solution trajectories lie on intersection



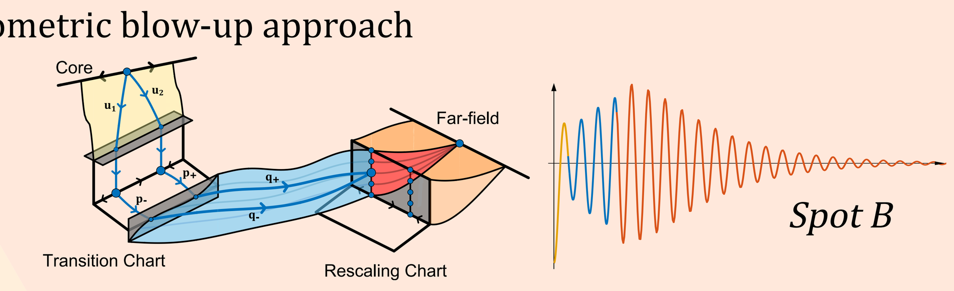
2009 - D.J.B. Lloyd & B. Sandstede, *Nonlinearity* [LS09]

Extension of [Sch03] to localised axisymmetric patterns



2013 - S.G. McCalla & B. Sandstede, *SIADS* [McS13]

Geometric blow-up approach



Polar Orthogonal Expansions

k-mode function: $\tilde{u}_k(\mathbf{x}) = \frac{1}{2\pi}\int_0^{2\pi} u(\mathbf{R}_\varphi \mathbf{x})\, e^{-ik\varphi}\, d\varphi$ (*2D rotation matrix*)

radial *k-coefficient*: $u_k(r) = \frac{1}{2\pi}\int_0^{2\pi} u(r\cos\varphi, r\sin\varphi)\, e^{-ik\varphi}\, d\varphi$

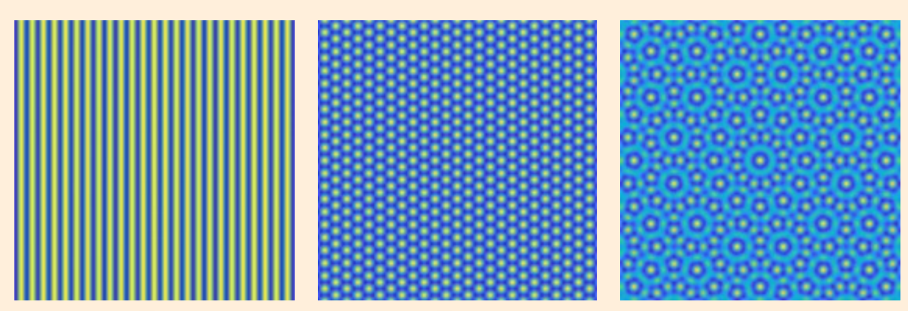
Expansions: $u(\mathbf{x}) = \sum_{n\in\mathbb{Z}} \tilde{u}_n(\mathbf{x})$ $\qquad u(r\cos\theta, r\sin\theta) = \sum_{n\in\mathbb{Z}} u_n(r)\, e^{in\theta}$

(1) becomes $\left\{ 0 = -(1+\Delta_n)u_n - \mu u_n + \nu \sum_{i+j=n} u_i u_j - \sum_{i+j+k=n} u_i u_j u_k \right\}_{n\in\mathbb{Z}}$

with n-Laplacian $\Delta_n = (\partial_r + \frac{1-n}{r})(\partial_r + \frac{n}{r})$ and continuity conditions $k\, u_k(0) = 0$

Reduced expansion for $\mathbb{D}_m$ dihedral functions: $u(r,\theta) = u(r,-\theta) = u(r, \theta + \frac{2\pi}{m})$ (*Invariant under discrete rotations*)

$$u(r,\theta) = \sum_{n\in\mathbb{Z}} u_{|n|}(r)\, \cos(mn\theta)$$



$\mathbb{D}_2$ $\qquad \mathbb{D}_6$ $\qquad \mathbb{D}_{12}$

Galerkin Approximation

Truncated expansion

$$u(r,\theta) = \sum_{|n|\leq N} u_{|n|}(r)\, \cos(mn\theta)$$

Project onto first (N+1) modes

Apply radial spatial dynamics to finite dimensional system

2023 - D.J.H, J.J. Bramburger & D.J.B. Lloyd, *Nonlinearity* [HBL23]

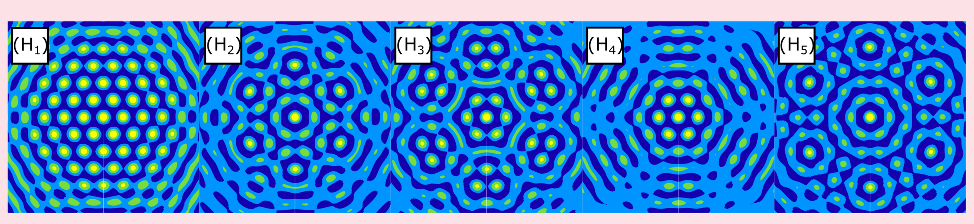
Any solution of the algebraic matching condition

$$a_n = 2\sum_{j=1}^{N-n} \cos(\tfrac{m\pi(n-j)}{3}) a_j a_{n+j} + \sum_{j=0}^{n} \cos(\tfrac{m\pi(n-2j)}{3}) a_j a_{n-j} \quad (2)$$

corresponds to a localised dihedral spot A pattern

2024 - D.J.H, J.J. Bramburger & D.J.B. Lloyd, *Nonlinearity* [HBL24]

Localised dihedral rings with a cubic matching condition



Some solutions to (2) for m=6, N=4 [HBL23]

Radial Vector Calculus

[Work in progress, upcoming paper]

Variables with physical representation (e.g., fluid velocity) require a choice of basis, which also depends on coordinates

Polar basis: $\mathbf{u}(r,\theta,z) = \sum_{n\in\mathbb{Z}} e^{in\theta}\{u_r^n(r,z)\,\hat{\mathbf{r}} + u_\theta^n(r,z)\,\hat{\boldsymbol{\theta}} + u_z^n(r,z)\,\hat{\mathbf{z}}\}$

$u(r,\theta,z) = \sum_{n\in\mathbb{Z}} e^{in\theta} u^n(r,z) \qquad \nabla u = \sum_{n\in\mathbb{Z}} e^{in\theta}\{\partial_r u^n\,\hat{\mathbf{r}} + \frac{in}{r}u^n\,\hat{\boldsymbol{\theta}} + \partial_z u^n\,\hat{\mathbf{z}}\}$

Complex basis: $\hat{\boldsymbol{\zeta}} = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) \qquad u_\zeta^k = \frac{1}{\sqrt{2}}(u_r^k - iu_\theta^k)$

$\hat{\bar{\boldsymbol{\zeta}}} = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} - i\hat{\mathbf{y}}) \qquad u_{\bar{\zeta}}^k = \frac{1}{\sqrt{2}}(u_r^k + iu_\theta^k)$

$$\mathbf{u} = \sum_{n\in\mathbb{Z}}\{e^{i(n-1)\theta} u_\zeta^n\,\hat{\boldsymbol{\zeta}} + e^{i(n+1)\theta} u_{\bar{\zeta}}^n\,\hat{\bar{\boldsymbol{\zeta}}} + e^{in\theta} u_z^n\,\hat{\mathbf{z}}\}$$

$$\nabla u = \sum_{n\in\mathbb{Z}}\{e^{i(n-1)\theta}\tfrac{1}{\sqrt{2}}\mathcal{D}_n u^n\,\hat{\boldsymbol{\zeta}} + e^{i(n+1)\theta}\tfrac{1}{\sqrt{2}}\mathcal{D}_{-n} u^n\,\hat{\bar{\boldsymbol{\zeta}}} + e^{in\theta}\partial_z u^n\,\hat{\mathbf{z}}\}$$

$\mathbf{u} \in (X(\mathbb{R}^3))^3 \implies$

$(u_\zeta^k, u_{\bar{\zeta}}^k, u_z^k) \in X_{(k-1)}([0,\infty)\times\mathbb{R}) \times X_{(k+1)}([0,\infty)\times\mathbb{R}) \times X_{(k)}([0,\infty)\times\mathbb{R})$

$$\Delta\mathbf{u} = \begin{vmatrix} \Delta_k u_r^k - \frac{1}{r^2}u_r^k - \frac{2ik}{r}u_\theta^k & \hat{\mathbf{r}} \\ \Delta_k u_\theta^k - \frac{1}{r^2}u_\theta^k + \frac{2ik}{r}u_r^k & \hat{\boldsymbol{\theta}} \\ \Delta_k u_z^k & \hat{\mathbf{z}} \end{vmatrix} = \begin{vmatrix} \Delta_{k-1}u_\zeta^k & \hat{\boldsymbol{\zeta}} \\ \Delta_{k+1}u_{\bar{\zeta}}^k & \hat{\bar{\boldsymbol{\zeta}}} \\ \Delta_k u_z^k & \hat{\mathbf{z}} \end{vmatrix}$$

Functions *á la mode*

2024 - M.D. Groves & D.J.H., *arXiv preprint* [GH24]

k-mode function and radial k-coefficient are related by

$$\tilde{u}_k(r\cos\theta, r\sin\theta) = u_k(r)\, e^{ik\theta}$$

Wirtinger derivatives: $\partial_\zeta = \frac{1}{\sqrt{2}}(\partial_x - i\partial_y) \qquad \partial_{\bar{\zeta}} = \frac{1}{\sqrt{2}}(\partial_x + i\partial_y)$

$\partial_\zeta \tilde{u}_k = \frac{1}{\sqrt{2}}\mathcal{D}_k u_k\, e^{i(k-1)\theta} \qquad \partial_{\bar{\zeta}}\tilde{u}_k = \frac{1}{\sqrt{2}}\mathcal{D}_{-k}u_k\, e^{i(k+1)\theta} \qquad \mathcal{D}_k = (\partial_r + \frac{k}{r})$

If u lies in some function space $X(\mathbb{R}^2)$, then $\tilde{u}_k$ lies in the proper subspace $\widetilde{X}_{(k)}(\mathbb{R}^2)$ which is isomorphic to the space $X_{(k)}([0,\infty))$ in which u_k lies.

e.g., $u \in C^m(\mathbb{R}^2) \implies \tilde{u}_k \in \widetilde{C}^m_{(k)}(\mathbb{R}^2) \iff u_k \in C^m_{(k)}([0,\infty))$

$u \in C^m(\mathbb{R}^2) \implies \begin{matrix} \partial_\zeta \tilde{u}_k \in \widetilde{C}^{m-1}_{(k-1)}(\mathbb{R}^2) \iff \mathcal{D}_k u_k \in C^{m-1}_{(k-1)}([0,\infty)) \\ \partial_{\bar{\zeta}}\tilde{u}_k \in \widetilde{C}^{m-1}_{(k+1)}(\mathbb{R}^2) \iff \mathcal{D}_{-k}u_k \in C^{m-1}_{(k+1)}([0,\infty)) \end{matrix}$

Fourier transform: $\mathcal{F}[u](\mathbf{k}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} u(\mathbf{x})\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x}$

Hankel transform: $\mathcal{H}_k[u_k](\rho) = \int_0^\infty u_k(r)\, J_k(\rho r)\, r\, dr$

$$\mathcal{F}[\tilde{u}_k](\rho\cos\omega, \rho\sin\omega) = i^{-k}\mathcal{H}_k[u_k](\rho)\, e^{ik\omega}$$

$$H^s_{(k)}([0,\infty)) = \{f \in L^2_1([0,\infty)) : (1+(\cdot)^2)^{\frac{s}{2}}\mathcal{H}_k[f] \in L^2_1([0,\infty))\}$$

$$L^2_1([0,\infty)) = L^2([0,\infty), r\, dr)$$

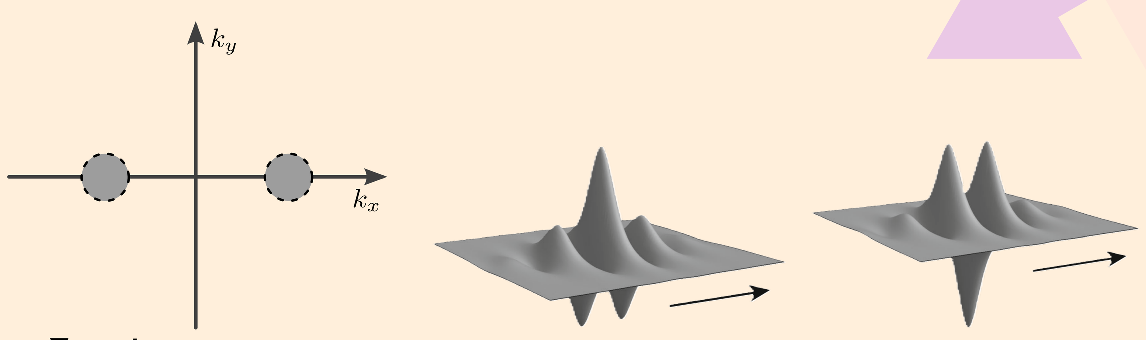
Pushing the Envelope

[Work in progress with M. Groves]

2018 - B. Buffoni et al., *ARMA* [BGW18]

Lyapunov-Schmidt proof of "fully localised" waves

Exponential decay in x and algebraic decay in y



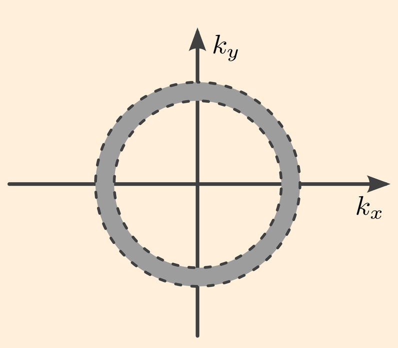
Fourier spectrum

Requires tools from functional analysis and modulation theory

What about fully localised planar patterns in (1)?

2024 - D.J.H. & D.J.B. Lloyd, *SIAM J. Appl. Math.* [HL24]

Formal derivation of radial envelope equations



Continuous Fourier spectrum

Multiple choice of patterns

Nonlinear resonances

Replace derivatives with Hankel multipliers

Extend existence proof to radial function spaces

Next Steps

Persistence result for Galerkin approximation

Fully localised patterns from Turing-Hopf bifurcations

Rigorous verification of envelope equations in [HL24]

Stability of localised planar structures (spots, patterns)

Planar invasion behaviour (spreading of pattern patches)

Localised 3D patterns - spherical harmonics expanions

$$u(r,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} u_{n,m}(r)\, Y_n^m(\theta,\varphi)$$

3D function spaces?

References

[LGR+15] [Sch03] [LS09] [McS13]

[HBL23] [HBL24] [GH24] [BGW18] [HL24]