Localised Spots on a Ferrofluid

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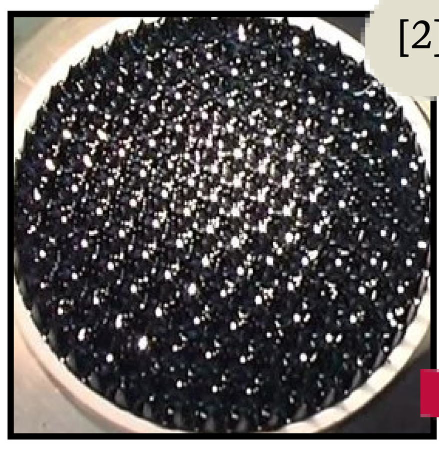
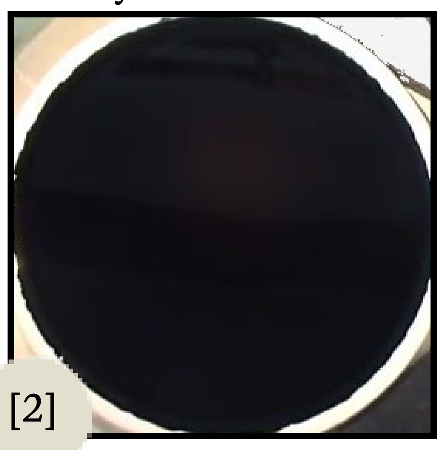
Rosensweig Instability


[2]

Ferrofluids are a colloidal suspension of iron nano-particles in a carrier fluid, resulting in a superparamagnetic fluid.

When subject to a strong magnetic field, the surface of the ferrofluid becomes unstable and spikes emerge, known as the 'Rosensweig instability'.


[2]

Significant progress has been made in analysing the Rosensweig instability for the case of doubly periodic structures, see [11].

One defining characteristic of the Rosensweig instability is the critical value of the applied field strength, where the spiked structures manifest throughout the domain.

Objective

Q: Is there a link between radial spot formation and the Rosensweig instability?

We aim to prove the existence of localised radial spots for the ferrofluid problem, and then determine the class of patterns from which radial spots bifurcate.

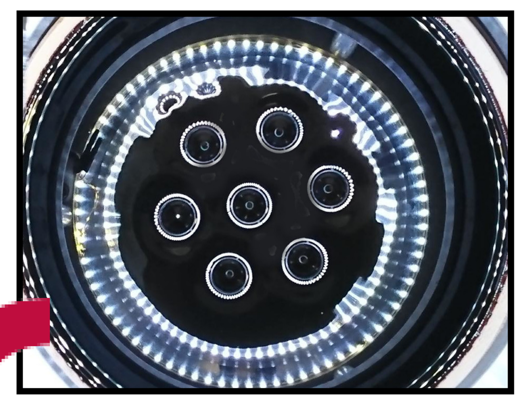

[3]

Smooth and bounded as $r \to 0$

Unaffected by shape of domain or boundary

Drifting -> Symmetry breaking

Localised Patterns



Localised radial spots have been seen experimentally [7], via a local perturbation of the magnetic field. However, these patterns persist even after this perturbation is removed, suggesting there may be an underlying mechanism for spots to spontaneously appear.

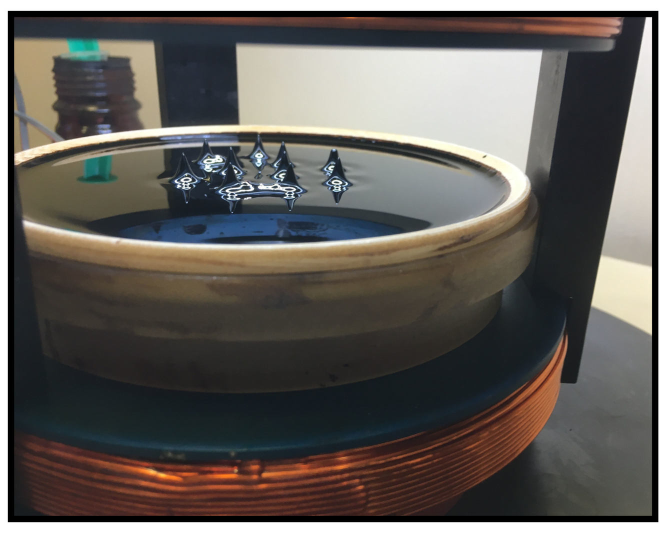
Localised patterns are employed in many areas of modelling; from vegetation in arid climates [8], to predicting and supressing crime 'hotspots' [10].

Gaining a better understanding of two-dimensional, and more specifically radial, localised patterns is one of the open problems presented in [4].


[1]

Set-Up

We assume the ferrofluid has a linear magnetisation law and is steady. We take cylindrical polar coordinates whose axes are centered at the spot, with finite depth and infinite radius. We also assume that the applied magnetic field is a constant uniform field in the z-direction.



Magnetic Potentials:

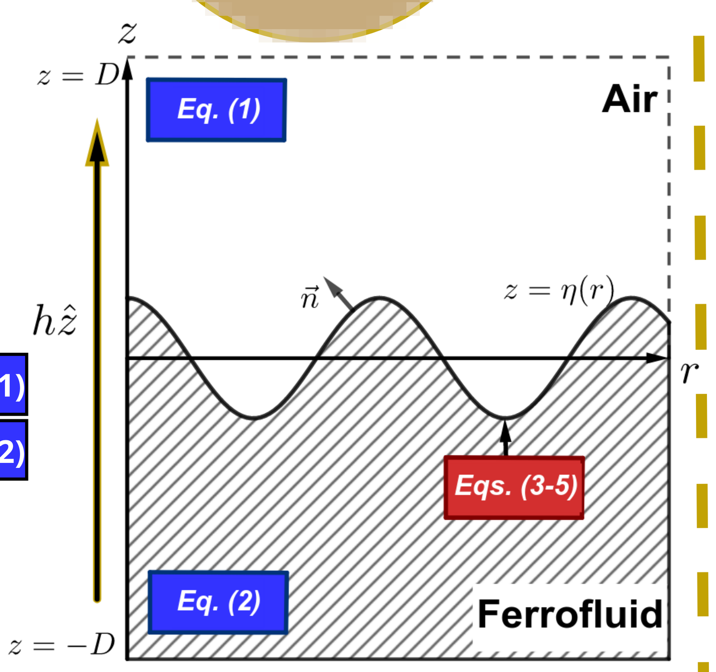
$$\begin{cases} \mathbf{H}_F = & -\nabla(\psi + hz), \\ \mathbf{H}_A = & -\nabla(\zeta + \mu h z), \end{cases}$$

Maxwell's equations:

$$\begin{cases} \nabla\cdot\mathbf{H} & = 0, \\ \nabla\times\mathbf{H} & = \mathbf{0}, \end{cases} \Longrightarrow \begin{cases} \Delta\zeta & = 0. \quad (1) \\ \Delta\psi & = 0. \quad (2) \end{cases}$$

Continuity equations at $z=\eta(r)$:

$$\begin{cases} \psi & = \zeta + (\mu-1)h\eta, \quad (3) \\ \mu(\psi_z - \eta_r\psi_r) & = (\zeta_z - \eta_r\zeta_r). \quad (4) \end{cases}$$



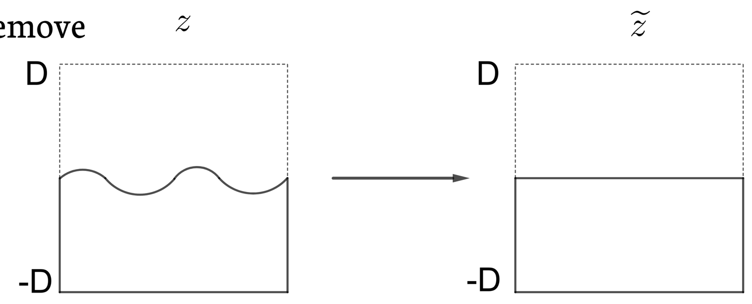
$$2\kappa = \nabla\cdot\hat{\mathbf{n}} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{-r\eta_r}{\sqrt{1+(\eta_r)^2}}\right)$$

Steady Euler + Ferrohydrostatic equations at $z=\eta(r)$:

$$2\kappa = \frac{\mu}{2}|\nabla\psi|^2 - \frac{1}{2}|\nabla\zeta|^2 + \mu(\psi_r\eta_r - \psi_z)\left[\psi_z - \zeta_z - (\mu-1)h\right] - \eta. \quad (5)$$

We use a flattening transformation to remove evolving domains,

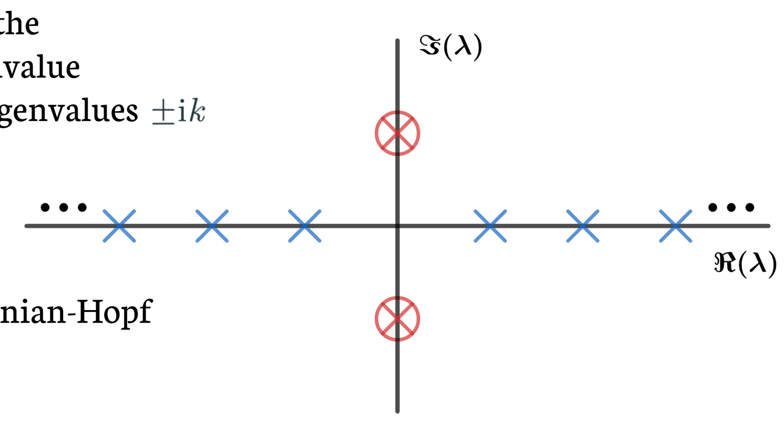
$$\tilde{z} := \begin{cases} \frac{z-\eta}{D-\eta}D, & \eta < z < D, \\ \frac{z-\eta}{D+\eta}D, & -D < z < \eta. \end{cases}$$



We introduce a bifurcation parameter ϵ, and construct a dynamical system,

$$\frac{\mathrm{d}}{\mathrm{d}r}\mathbf{u}(r,z) = \mathbf{L}\left(\frac{1}{r}\right)\mathbf{u}(r,z) + \mathbf{F}(\mathbf{u},r,\epsilon)$$

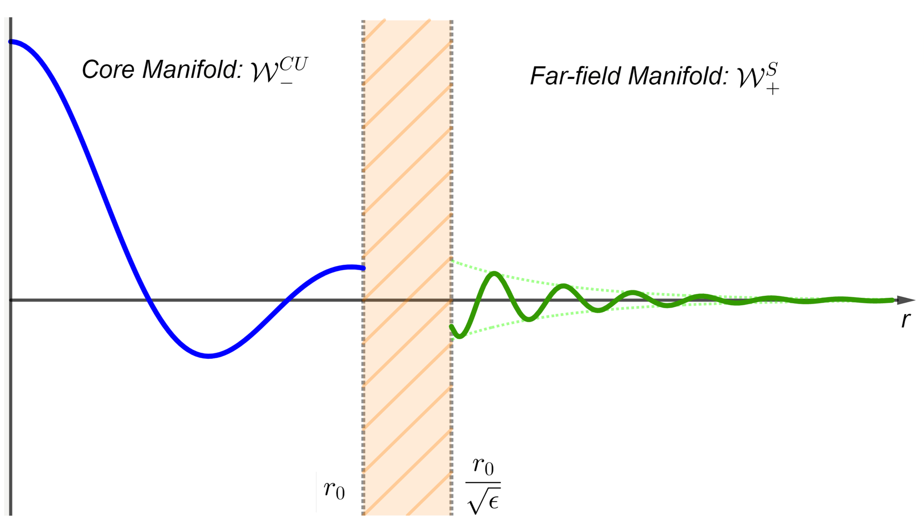
We look for separable solutions at the bifurcation point, solving the eigenvalue problem for $\mathbf{L}(0)(r\to\infty)$, with eigenvalues $\pm ik$ (repeated), and $\{\pm\lambda_j\}_{j=1}^{\infty} \in \mathbb{R}$.



As ϵ increases, we have a Hamiltonian-Hopf bifurcation.

Manifolds

We construct local solutions as $r\to 0$ and $r\to\infty$, respectively. To do this, we employ radial centre-manifold reduction thoeory of Scheel [9]. We isolate the core manifold, containing all solutions which are bounded as $r\to 0$, and the far-field manifold, containing all solutions which decay exponentially as $r\to\infty$.



Core solutions are spanned by:

- $J_0(kr)$ (Spots)
- $rJ_1(kr)$ (Rings)

We write down a variation of constants formula:

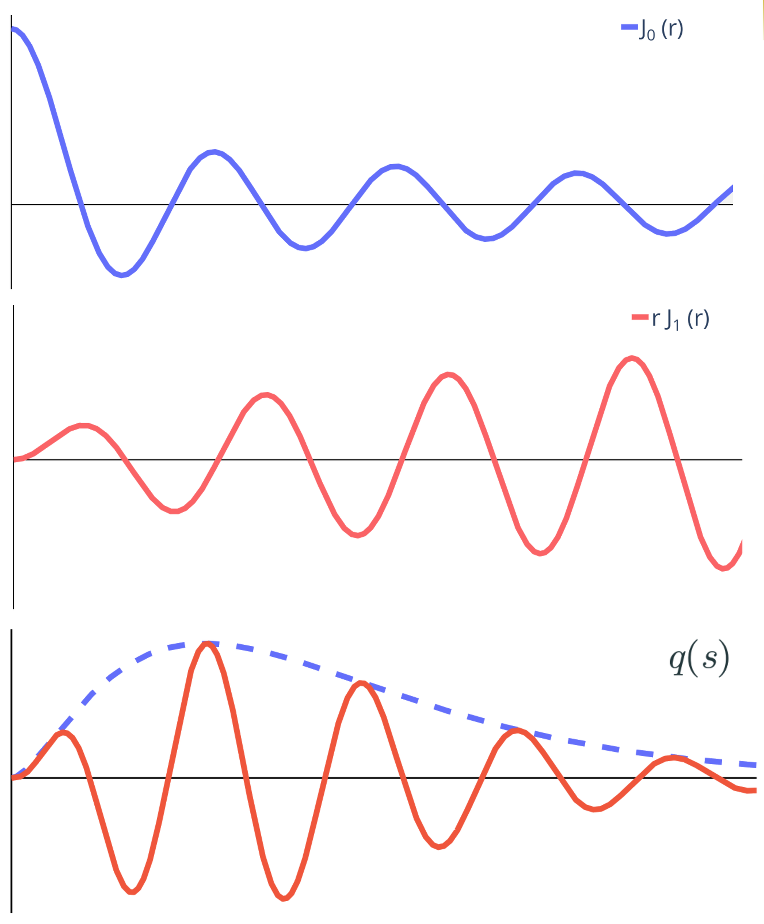
$$\mathbf{u}(r) = \Phi(r)\Phi^{-1}(0)\mathbf{u}_0 + \int_0^r \Phi(r-s)\mathbf{F}(\mathbf{u}(s))\mathrm{d}s.$$

and parameterise the core manifold.

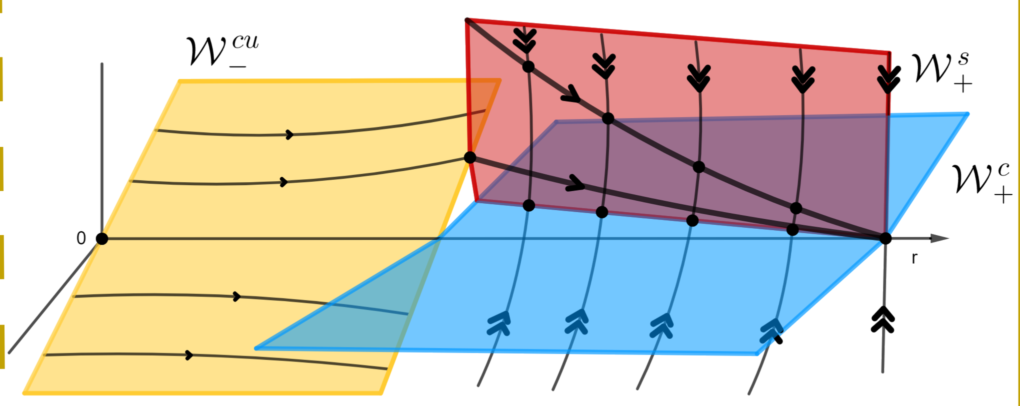
For the far-field problem, we define $\sigma := \frac{1}{r}$ and extend the system.

Working in slow coordinates $s := r\sqrt{\epsilon}$, far-field solutions are constrained by some envelope function $q(s)$:

$$q_{ss} + \frac{1}{s}q_s - \frac{1}{s^2}q = q - c|q|^2q + \mathrm{O}(|q|^4)$$

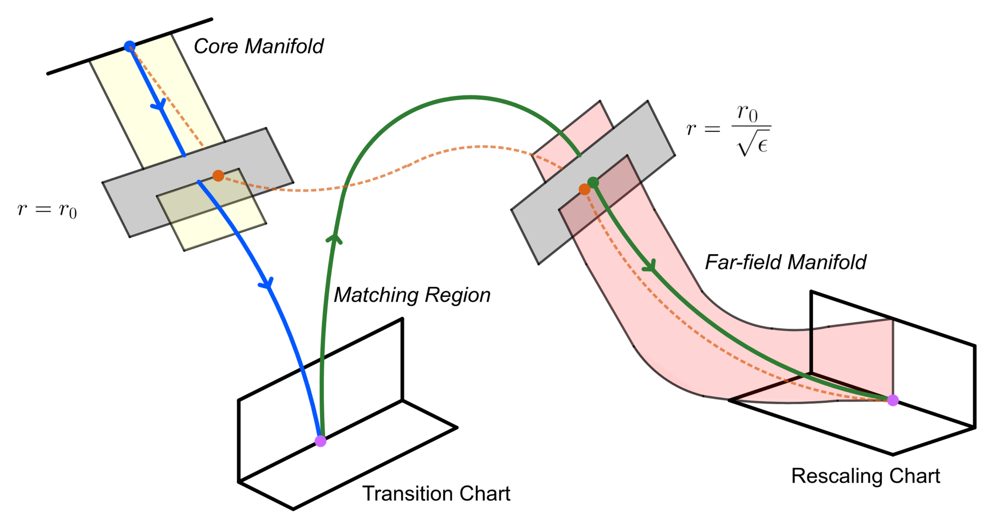


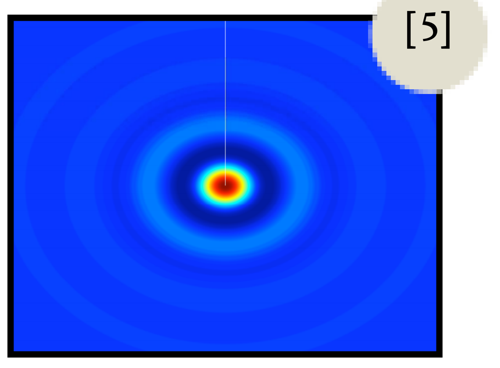
To parameterise the far-field stable manifold we first construct the centre-stable manifold, containing all solutions bounded as $r\to\infty$. We then introduce smooth foliations, as seen in [6], so that we can define the stable manifold as a set of mappings from the centre-stable manifold to the centre manifold (as pictured below)



Matching

In order to reconcile the algebraic behaviour of the core solutions with the exponential decay in the far-field, we employ Geometric Blow-Up methods, as seen in [6].




[5]

When matching on the centre-manifold, the amplitude equations of the system are equivalent to the Swift-Hohenberg equation, up to leading order.

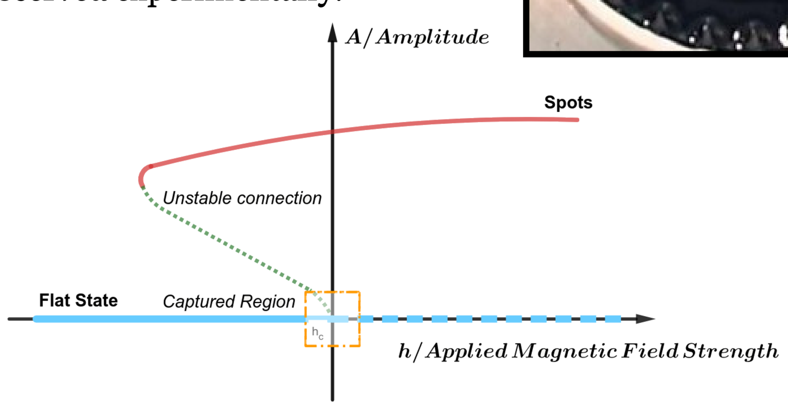
Existence of radial spots have been proven for the Swift-Hohenberg equation in [5] and [6].

Future Work

After proving the existence of radial spots, we want to use linear stability techniques to capture the dynamics near the bifurcation point.


[2]

Another area of interest is the interaction between 2 localised spots, which have been observed experimentally.



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